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Local Modifications of Damped Linear Systems

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A procedure is developed for determining the eigenvalues and eigenvectors of a discrete linear-vibration system resulting from the addition or removal of a discrete element. In this procedure the known characteristics of the original system are used to generate the modified characteristic equation directly without having to solve the modified eigenvalue problem explicitly. Because of the form of the modified characteristic equation, the problem is ideally suited to numerical solution by the Newton Raphson iteration procedure. If repeated eigenvalues exist, the system matrices may not be diagonalizable by classical modal methods. However, the system can be reduced to Jordan Canonical form and the procedure presented incorporates this possibility.

Introduction

ANALYTICAL investigations of dynamic elastic systems frequently involve a determination of the effect of a change in a particular system component on the natural frequencies and normal modes of vibration. Often, a change of this type is not sufficiently small to permit a solution by standard perturbation techniques, and alternative procedures are required. In an investigation of some problems associated with the addition of localized springs and masses to undamped one-dimensional continuous systems, Weissenburger[1] developed a method of analysis called "eigenvalue modification" which is applicable for both small and large modifications. This procedure utilizes an eigenfunction expansion to express the eigenfunctions (normal modes) of the modified system in terms of the known eigenfunctions of the unmodified system; but since the actual computation is performed with only a finite number of terms, the problem is equivalent to an investigation of a local modification of a symmetric positive definite matrix eigenvalue problem. Although the use of an eigenfunction expansion to solve such problems is in no way unique, his method of solving for the frequencies and modes associated with such a modification is much simpler and more accurate than previous procedures.[2]

In this paper, Weissenburger's procedure is recast into matrix form for direct application to discrete vibration systems. It is also extended to include the effects of linear viscous damping, both in the original system and in the modification, with the result that the eigenvalues may be complex quantities, and the eigenvectors associated with multiple eigenvalues may not form a spanning set for the vector space. In addition, the restriction to symmetric sign-definite‡ matrices, which was an implicit requirement in Weissenburger's work with self-adjoint positive-definite systems, is relaxed in this development and the simplifications which arise as a result of distinct eigenvalues and symmetric positive-definite matrices are indicated.

Derivation of Equations

Original System

In the solution of the modified eigenvalue problem arising from a localized change in an n degree of freedom vibration system, a complete knowledge of the original system eigenvalues and eigenvectors is required. With the inclusion of linear-viscous damping, these original characteristics are assumed to be those associated with the equivalent $2n$ dimensional matrix system:

$$\left(\mu\left[\begin{array}{c|c}[0] & [m]\\ \hline [m] & [c]\end{array}\right] + \left[\begin{array}{c|c}-[m] & [0]\\ \hline [0] & [k]\end{array}\right]\right)\left\{\begin{array}{c}\mu\{X\}\\ \hline \{X\}\end{array}\right\} = \{0\} \quad (1)$$

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‡ In this paper, the term sign-definite refers to either positive definite or non-negative definite.

obtained by adjoining the identify

$$\mu[m]\{X\} - \mu[m]\{X\} = \{0\} \tag{2}$$

to the usual n dimensional system

$$[\mu^2[m] + \mu[c] + [k]]\{X\} = \{0\} \tag{3}$$

Although in most applications the mass matrix $[m]$, the damping matrix $[c]$, and the stiffness matrix $[k]$ are real symmetric sign-definite matrices, these properties are not needed for the development which follows; it is only necessary that $[m]$ be nonsingular.

For convenience in the remainder of the development, Eq. (1) will be rewritten as

$$[\mu[M] + [K]]\{\Phi\} = \{0\} \tag{4}$$

by introducing the following definitions

$$[M] = \left[\begin{array}{c|c} [0] & [m] \\ \hline [m] & [c] \end{array}\right] \quad [K] = \left[\begin{array}{c|c} -[m] & [0] \\ \hline [0] & [k] \end{array}\right]$$

$$\{\Phi\} = \left\{\begin{array}{c} \mu\{X\} \\ \hline \{X\} \end{array}\right\}$$

Furthermore, since $[m]$ is nonsingular, $[M]^{-1}$ exists, and with the definition of the dynamical matrix $[D] = -[M]^{-1}[K]$, Eq. (4) may be reduced to the structure of a standard matrix eigenvalue problem

$$[\mu\lceil I \rfloor - [D]]\{\Phi\} = \{0\} \tag{5}$$

The complete solution of Eq. (5) consists of $2n$ eigenvalues $\mu_r(r = 1, 2, \ldots, 2n)$ and the modal matrix $[\Phi]$, the columns of which are the eigenvectors $\{\Phi\}_r$ of $[D]$. For any two distinct eigenvalues μ_r and μ_s, the corresponding eigenvectors can be shown to satisfy the orthogonality condition

$$\lfloor \zeta \rfloor_r [D]\{\Phi\}_s = \delta_{rs}\mu_r \tag{6}$$

where δ_{rs} is the Kronecker delta, $\{\Phi\}_s$ is the sth column of $[\Phi]$, and $\lfloor \zeta \rfloor_r$ is the rth row of the matrix $[\Phi]^{-1}$, also known as a left eigenvector§ of $[D]$ since it is the nontrivial-row vector $\lfloor \zeta \rfloor$ which satisfies

$$\lfloor \zeta \rfloor [\mu_r \lceil I \rfloor - [D]] = \lfloor 0 \rfloor \tag{7}$$

For the case of multiple eigenvalues, that is, when $\mu_r = \mu_s$ for at least one pair $r \neq s$, there may not be $2n$ linearly independent eigenvectors to form $[\Phi]$, that is, the eigenvectors may not form a spanning set for the vector space, and the resulting modal matrix is singular. In this case, for each set of multiple eigenvalues, the missing ordinary eigenvectors are replaced by linear combinations of generalized eigenvectors of rank t defined by the pair of relationships

$$[\mu_r\lceil I \rfloor - [D]]^t\{\Phi\} = \{0\}$$

$$[\mu_r\lceil I \rfloor - [D]]^{t-1}\{\Phi\} \neq \{0\}$$

The index t, known as the rank of the generalized eigenvector, is related to the multiplicity of the eigenvalue.[3]

Proceeding in the manner described above, a linearly independent set of vectors $\{\Phi\}_r$ may be obtained, and the modal matrix $[\Phi]$ thus formed and its inverse $[\Phi]^{-1}$ may be used to reduce the original system to Jordan canonical form by the similarity transformation

$$[\Phi]^{-1}[D][\Phi] = [J] \tag{8}$$

The matrix $[J]$ in Eq. (8), called a Jordan matrix, is a block diagonal matrix of the form

$$[J] = \begin{bmatrix} [J_1] & & & & & \\ & [J_2] & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & [J_k] & \\ & & & & & \cdot \\ & & & & & & \cdot \end{bmatrix} \tag{9}$$

in which the kth block $[J_k]$,

$$[J_k] = \begin{bmatrix} \mu & 1 & 0 & \ldots & & 0 \\ 0 & \mu & 1 & & & \cdot \\ & & & & & \cdot \\ & & & & & \cdot \\ 0 & 0 & \mu & & 1 & 0 \\ \cdot & \cdot & & \cdot & & \\ \cdot & \cdot & & & \cdot & 1 \\ \cdot & \cdot & & & \cdot & 1 \\ 0 & 0 & \ldots & & 0 & x \end{bmatrix} \tag{10}$$

called a Jordan block of dimension m_k, has the same eigenvalue μ on the principal diagonal, ones on the superdiagonal, and zeroes elsewhere. There is one Jordan block for each ordinary eigenvector used to form $[\Phi]$. Thus, if no generalized eigenvectors were required to form $[\Phi]$, there would be $2n$ Jordan blocks, each of dimension 1, and $[J]$ would be a strictly diagonal matrix with the eigenvalues on the principal diagonal.

Modified System

The vibration system just considered could be modified locally by introducing a change in one of the system springs, dampers, or masses. Consider the system in Eq. (1) to be modified by a change in the stiffness of a linear spring. A modification of this type may be expressed by the addition of the matrix $\alpha\{p\}\lfloor q \rfloor$ to $[k]$, where α is the magnitude of the stiffness change, and the column-row product $\{p\}\lfloor q \rfloor$ designates the location of the modification. In $2n$ space the corresponding eigenvalue problem has the form

$$[\lambda\lceil I \rfloor - [D] + \alpha\{P\}\lfloor Q \rfloor]\{Y\} = \{0\} \tag{11}$$

where the modified eigenvalue is represented by λ, the modified eigenvector by $\{Y\}$, and the transformed modification vectors by

$$\{P\} = [M]^{-1}\left\{\begin{array}{c} \{0\} \\ \hline \{p\} \end{array}\right\} \quad \text{and} \quad \lfloor Q \rfloor = \lfloor \lfloor 0 \rfloor \,\vdots\, \lfloor q \rfloor \rfloor$$

By letting $[B] = [D] - \alpha\{P\}\lfloor Q \rfloor$, the solution of the modified eigenvalue problem could be obtained by the same procedure that was used to solve the original system. However, because of the knowledge of the original system characteristics and because of the column-row product form of the modification matrix, the $2n$ equations in Eq. (11) satisfy an additional common relationship which may be used to simplify the solution.

To demonstrate this simplification, Eq. (11) is transformed by first introducing the coordinate transformation $\{Y\} = [\Phi]\{Z\}$ and then premultiplying the result by $[\Phi]^{-1}$ to obtain

$$[\lambda\lceil I \rfloor - [J] + \alpha\{U\}\lfloor V \rfloor]\{Z\} = \{0\} \tag{12}$$

where $\{U\} = [\Phi]^{-1}\{P\}$, $\lfloor V \rfloor = \lfloor Q \rfloor[\Phi]$, and $[J]$ is the Jordan matrix of the original system described in Eq. (8). Since the Jordan blocks may be rearranged along the principal diagonal of $[J]$ by a simple reordering of the columns of $[\Phi]$ and the rows of $[\Phi]^{-1}$, it can be assumed without restriction that the first Jordan block $[J_1]$ is the Jordan block of greatest dimension and that $m_1 \geqslant 1$.

If $m_1 = 1$, that is, if $[J]$ is strictly diagonal, the $2n$ equations represented by Eq. (12) are uncoupled and a representative

§ The designation of $\lfloor \zeta \rfloor$ as a left eigenvector is to distinguish it from the more common right eigenvector $\{\Phi\}$ defined by Eq. (5).

equation may be written as

$$(\lambda - \mu_r)Z_r + \alpha U_r \sum_{k=1}^{2n} V_k Z_k = 0 \tag{13}$$

If, however, $m_1 > 1$, it can be seen from the form of $[J]$ that the m_1st Eq. in (12)

$$(\lambda - \mu_{m_1})Z_{m_1} + \alpha U_{m_1} \sum_{k=1}^{2n} V_k Z_k = 0 \tag{14}$$

is uncoupled from the remaining equations, while the $(m_1 - 1)$st equation is coupled with the m_1st and has the form

$$(\lambda - \mu_{m_1-1})Z_{m_1-1} - Z_{m_1} + \alpha U_{m_1-1} \sum_{k=1}^{2n} V_k Z_k = 0 \tag{15}$$

but if Eq. (14) is introduced into Eq. (15), the following equation results

$$(\lambda - \mu_{m_1-1})Z_{m_1-1} + \alpha \left(U_{m_1-1} + \frac{U_{m_1}}{(\lambda - \mu_{m_1})} \right) \times \sum_{k=1}^{2n} V_k Z_k = 0$$

By defining

$$U_{m_1}{}^* = U_{m_1}$$
$$U_{m_1-1}{}^* = U_{m_1-1} + U_{m_1}{}^*/(\lambda - \mu_{m_1})$$
$$\vdots$$
$$U_1{}^* = U_1 + U_2{}^*/(\lambda - \mu_2) \tag{16}$$

and by extending this defining procedure to all other coupled sets of equations corresponding to the remaining Jordan blocks, a representative equation

$$(\lambda - \mu_r)Z_r + \alpha U_r{}^* \sum_{k=1}^{2n} V_k Z_k = 0 \tag{17}$$

where $r = 1, 2, \ldots 2n$, which is similar to Eq. (13), may be formed.

Each $U_r{}^*$ in Eq. (17) which equals zero allows a simplification in the solution procedure that will be described in the section Numerical Considerations. Therefore, for the more general case in which each $U_r{}^* \neq 0$, the set of equations in Eq. (17) may be rewritten as

$$(\lambda - \mu_1)Z_1/U_1{}^* = \ldots = (\lambda - \mu_{2n})Z_{2n}/U_{2n}{}^* = \alpha \sum_{k=1}^{2n} V_k Z_k \tag{18}$$

or simply

$$Z_k = [(\lambda - \mu_r)Z_r/U_r{}^*][U_k{}^*/(\lambda - \mu_k)] \tag{19}$$

Upon substitution of Eq. (19) into Eq. (17), the following equation results

$$0 = \frac{1}{\alpha} + \sum_{k=1}^{2n} \frac{U_k{}^* V_k}{(\lambda - \mu_k)} \tag{20}$$

Equation (20) is the characteristic equation of the modified system since its $2n$ roots $\lambda_r (r = 1, 2, \ldots, 2n)$ are the eigenvalues of the modified system. Corresponding to each of the eigenvalues λ_r, the $2n$ elements Z_{kr} of a corresponding transformed eigenvector $\{Z\}_r$ may be generated from Eq. (19), that is,

$$Z_{kr} = C_r \frac{U_k{}^*}{(\lambda_r - \mu_k)} \tag{21}$$

By means of the transformation $\{Y\} = [\Phi]\{Z\}$, the modified eigenvectors $\{Y\}_r$ of the system are obtained. Because multiple eigenvalues generate identical vectors $\{Z\}$, alternative procedures must be used to generate generalized eigenvectors associated with this case.[4]

The development of Eqs. (11–21) demonstrates a procedure for incorporating stiffness modifications in a damped vibration system. Damping modifications can be incorporated in an analogous manner by reformulating the problem in terms of eigenvalue reciprocals. This procedure utilizes a knowledge of the original system characteristics obtained from

$$[D] = [K]^{-1}[M]$$

which may be obtained when both $[k]$ and $[m]$ are nonsingular. This imposes no serious restrictions; however, since singular stiffness matrices arise as a result of zero valued eigenvalues, and these eigenvalues, along with their associated rigid body modes, may be removed by constraint equations prior to solution.

Because of the multiple appearance of $[m]$ in $[M]$ and $[K]$, mass modifications require three successive applications of a procedure similar to that just described, and although a formal solution has been obtained,[4] the practical value of its use may be questionable.

Symmetric Positive-Definite Systems

The modification procedure admits to significant simplifications when the original system matrices and the modification matrix are real symmetric and positive definite (actually, the matrices $[c]$, $[k]$, and $\alpha\{p\}\lfloor p\rfloor$ may be non-negative definite) and the original system is characterized by distinct eigenvalues. In this case, the original system in Eq. (4) may be reduced to diagonal form directly by use of the orthogonality conditions

$$[\Phi]^T[M][\Phi] = \lceil I \rfloor, \quad [\Phi]^T[K][\Phi] = -\lceil \mu \rfloor$$

presented by Foss.[5] The corresponding symmetric form of the modified characteristic equation then has the simple structure

$$0 = \frac{1}{\alpha} + \sum_{k=1}^{2n} \frac{U_k{}^2}{(\lambda - \mu_k)} \tag{22}$$

where

$$\{U\} = [\Phi]^T \begin{Bmatrix} \{0\} \\ \{p\} \end{Bmatrix}$$

This form of the solution, when applicable, is much easier to apply since the transpose, not the inverse, of the original modal matrix is required, and the coefficients U_k are not functions of the eigenvalue λ.

Numerical Considerations

Unchanged Eigenvalues

The modified characteristic Eq. (20) was obtained by imposing the restriction that each $U_r{}^* \neq 0$, but it was indicated that simplifications occur when one or more of the elements $U_r{}^* = 0$. This simplification will now be presented.

Consider the set of equations represented by Eq. (17) which may be expressed in the matrix form

$$\begin{bmatrix} (\lambda - \mu_1) + \alpha U_1^* V_1 & \cdots & \alpha U_1^* V_k & \cdots & \alpha U_1^* V_{2n} \\ \alpha U_2^* V_1 & \cdots & \alpha U_2^* V_k & \cdots & \alpha U_2^* V_{2n} \\ \vdots & & \vdots & & \vdots \\ \alpha U_k^* V_1 & \cdots & (\lambda - \mu_k) + \alpha U_k^* V_k & \cdots & \alpha U_k^* V_{2n} \\ \vdots & & \vdots & & \vdots \\ \alpha U_{2n}^* V_1 & \cdots & \alpha U_{2n}^* V_k & \cdots & (\lambda - \mu_{2n}) + \alpha U_{2n}^* V_{2n} \end{bmatrix} \{Z\} = \{0\}$$

If a particular modification coefficient U_k^* equals zero, then the kth row of the coefficient matrix will have all zero valued elements except for the term on the principal diagonal, and this element will be $(\lambda - \mu_k)$. Since the characteristic Eq. (20) is equivalent to the equation obtained, by setting the determinant of the coefficient matrix equal to zero, it can be seen, by considering an expansion of the determinant on the elements of the kth row, that $(\lambda - \mu_k)$ is a factor of the characteristic equation, and $\lambda = \mu_k$ is therefore a root. Thus, a sufficient condition for the existence of an eigenvalue μ_k which remains unchanged by a local modification is that $U_k^* = 0$.

Having thus found a root $\lambda = \mu_k$, the kth equation in Eq. (18) may be omitted, and the remaining $2n - 1$ equations may be combined to form the modified characteristic equation, the roots of which will be the remaining $2n - 1$ eigenvalues of the modified system. Identical procedures may be used to simplify the solution further when more than one unchanged eigenvalue exists.

Since the eigenvalues of the system in Eq. (12) are identical to those of the transposed system

$$[\lambda \lceil I \rfloor - [J]^T + \alpha\{V\}\lfloor U\rfloor]\{Z\} = \{0\} \tag{23}$$

another sufficient condition for the existence of an unchanged eigenvalue is that $V_k^* = 0$, where for the first Jordan block the elements V_k^* are defined by

$$\begin{aligned} V_1^* &= V_1 \\ V_2^* &= V_1^*/(\lambda - \mu_1) + V_2 \\ &\;\,\vdots \\ V_{m_1}^* &= V_{m-1}^*/(\lambda - \mu_{m-1}) + V_{m_1} \end{aligned} \tag{24}$$

In general, the unchanged eigenvalues obtained from a consideration of the transposed system will not be the same as those indicated by zero valued elements U_k^*. It can be seen that the existence of unchanged eigenvalues simplifies the solution of the modified eigenvalue problem, since it permits the solution of a smaller order system.

Solution of the Modified Characteristic Equation

The most important feature of this modification procedure is the ease with which the new characteristic equation can be formed from the known solution of the original system. The value of the method, however, is further enhanced by the fact that the roots of this equation may be obtained by a direct application of existing numerical techniques; the derivative of the characteristic equation is easily obtained and accurate starting values may be generated, so the Newton-Raphson iteration procedure is a logical technique to use. This is particularly true for the symmetric, positive-definite case with distinct eigenvalues.

In the Newton-Raphson procedure an approximate zero $\lambda_{(s)}$ of a function $f(\lambda)$ is used to generate an improved approximation $\lambda_{(s+1)}$ by means of the relationship

$$\lambda_{(s+1)} = \lambda_{(s)} - f(\lambda_{(s)})/f'(\lambda_{(s)}) \tag{25}$$

Successive applications of Eq. (25), beginning with an initial estimate $\lambda_{(0)}$, yields a sequence of iterates, which if convergent, approaches a zero of the function $f(\lambda)$. For values of $\lambda_{(s)}$ in the vicinity of a simple zero, convergence is known to be quadratic; convergence is only linear for multiple zeroes, however.

To apply the Newton-Raphson procedure to the class of modification problems having distinct eigenvalues, $f(\lambda)$ is given by

$$f(\lambda) = \frac{1}{\alpha} + \sum_{k=1}^{2n} \frac{U_k^2}{(\lambda - \mu_k)} \tag{26}$$

and the first derivative, which is also required, is

$$f'(\lambda) = - \sum_{k=1}^{2n} \frac{U_k^2}{(\lambda - \mu_k)^2} \tag{27}$$

It is the similarity of form between $f(\lambda)$ and $f'(\lambda)$ which makes the solution convenient; the evaluation of $f'(\lambda)$ and $f(\lambda)$ may be carried out with only slightly more effort than is required for an evaluation of $f(\lambda)$ alone.

To begin the iterative scheme in the solution of the rth zero of $f(\lambda)$, a starting value or initial estimate of λ_r is needed. This may be obtained by writing

$$\lambda_r = \mu_r + \epsilon \tag{28}$$

where ϵ represents the change in the rth eigenvalue. A substitution of Eq. (28) into (26) yields

$$0 = f(\lambda_r) = \frac{1}{\alpha} + \frac{U_r^2}{\epsilon} + \sum_{k \neq r}^{2n} \frac{U_k^2}{(\mu_r - \mu_k) + \epsilon} \tag{29}$$

When the eigenvalues of the original system are sufficiently separated and the modification is small, the summation in Eq. (29) is negligible and a reasonable value of ϵ

$$\epsilon = -\alpha U_r^2$$

may be obtained. For a more refined estimate each term in the summation may be expanded in the series

$$1/[(\mu_r - \mu_k) + \epsilon] = [1/(\mu_r - \mu_k)][1 - \epsilon/(\mu_r - \mu_k) + \epsilon^2/(\mu_r - \mu_k)^2 \ldots] \tag{30}$$

and a two-term approximation of Eq. (30), when substituted into (29) yields¶

$$\epsilon = \{(S_1 + 1/\alpha) \pm [(S_1 + 1/\alpha)^2 + 4S_2U_r^2]^{1/2}\}/2S_2$$

where

$$S_1 = \sum_{k \neq r}^{2n} \frac{U_k^2}{(\mu_r - \mu_k)} \quad \text{and} \quad S_2 = \sum_{k \neq r}^{2n} \frac{U_k^2}{(\mu_r - \mu_k)^2}$$

Although the one-term approximation is sufficiently accurate for most applications, the two-term estimate requires only about as much computation as the performance of one iteration and is usually worth the effort.

The accuracy of the modified eigenvalues obtained by this procedure is dependent only upon the accuracy of the known

¶ This estimate is analogous to that obtained by Weissenburger,[1] but here all quantities may be complex as a result of the inclusion of viscous damping.

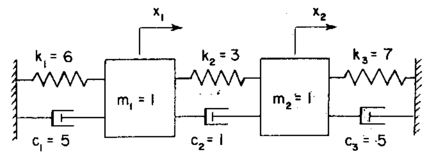

Fig. 1 Damped spring-mass system.

original solution and the strictness of the convergence requirement selected for the iteration procedure. An additional feature also exists: when the initial estimate for the Newton-Raphson procedure is sufficiently accurate, there is no need to suppress known roots of the characteristic equation so that cumulative errors are not encountered in the higher eigenvalues.

In the experience of the authors, the two-term approximation for generating the starting value for symmetric positive-definite systems having distinct eigenvalues has always been adequate, and no numerical difficulties have been encountered. For systems with multiple eigenvalues, however, the numbers U_k^* are functions of λ, so the derivative $f'(\lambda)$ does not have the simple form of Eq. (27). For these cases, the individual iterations involve more computations and convergence is slower.

Numerical Examples

To demonstrate the use of the modification procedure, two numerical examples will be presented. In the first, a system having multiple eigenvalues is considered, while the second demonstrates an application of the special case, namely, a symmetric positive-definite system with distinct eigenvalues. These examples have not been designed to test the limits of the procedure; rather, the systems have been kept relatively simple so that the results would not be masked by an inordinate recording of numerical data. Actually, the modification procedure is most beneficial when applied to large systems.

Example 1

As a demonstration of the general modification procedure for systems having multiple eigenvalues, the damped spring-mass system in Fig. 1, which has eigenvalues $\mu_1 = \mu_2 = \mu_3 = \mu_4 = -3$, will be modified by effecting a unit decrease of the linear spring k_3. Although the original system matrices

$$[m] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad [c] = \begin{bmatrix} 6 & -1 \\ -1 & 6 \end{bmatrix} \quad [k] = \begin{bmatrix} 9 & -3 \\ -3 & 10 \end{bmatrix}$$

are symmetric and positive definite, no use is made of these properties. Corresponding to the four equal eigenvalues, there are one ordinary and three generalized eigenvectors which are used to form the modal matrix

$$[\Phi] = \begin{bmatrix} 3 & -10 & 3 & 0 \\ 0 & 3 & -10 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & -1 & 3 & 1 \end{bmatrix}$$

of the original system. This matrix, together with its inverse

$$[\Phi]^{-1} = \begin{bmatrix} -30 & -9 & -91 & 0 \\ -10 & -3 & -30 & 0 \\ -3 & -1 & -9 & 0 \\ -1 & 0 & -3 & 1 \end{bmatrix}$$

and the original equal eigenvalues constitute the known solution of the given system.

The stiffness modification indicated previously may be represented by the addition of the matrix $\alpha\{p\}\lfloor q \rfloor$ to $[k]$, where

$$\alpha = -1, p = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}, \text{ and } \lfloor q \rfloor = \lfloor 0 \quad 1 \rfloor$$

Thus, from these vectors and the characteristics of the original system, the transformed augmented vectors

$$\{U\} = [\Phi]^{-1}[M]^{-1} \begin{Bmatrix} \{0\} \\ \{p\} \end{Bmatrix} = \begin{Bmatrix} -9 \\ -3 \\ -1 \\ 0 \end{Bmatrix}$$

and $\lfloor V \rfloor = \lfloor \lfloor 0 \rfloor \vdots \lfloor q \rfloor \rfloor [\Phi] = \lfloor 0 \quad -1 \quad 3 \quad 1 \rfloor$ are generated, and the effective modification coefficients defined by Eqs. (16) and (24)

$$U_4^* = 0, \quad U_3^* = -1, \quad U_2^* = -1/(\lambda + 3) - 3$$

$$U_1^* = -1/(\lambda + 3)^2 - 3/(\lambda + 3) - 9$$

$$V_1^* = 0, \quad V_2^* = -1, \quad V_3^* = -1/(\lambda + 3) + 3$$

$$V_4^* = -1/(\lambda + 3)^2 + 3/(\lambda + 3) + 1$$

may be formed. Since U_4^* and V_1^* are both equal to zero, there are two unchanged eigenvalues $\lambda_1 = -3$ and $\lambda_2 = -3$ in the modified system, and the corresponding eigenvectors are found to be

$$\{Z\}_1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad \{Z\}_2 = \begin{Bmatrix} 0 \\ 9 \\ 3 \\ 1 \end{Bmatrix}$$

The two remaining eigenvalues $\lambda_3 = -2$ and $\lambda_4 = -4$ are the two roots of the reduced characteristic Eq. (20), and the associated eigenvectors are found, by use of Eq. (21) to be

$$\{Z\}_3 = \begin{Bmatrix} 13 \\ 4 \\ 1 \\ 0 \end{Bmatrix} \quad \{Z\}_4 = \begin{Bmatrix} 7 \\ 2 \\ 1 \\ 0 \end{Bmatrix}$$

These four vectors become the column entries in the matrix

$$[Z] = \begin{bmatrix} 0 & 1 & 13 & 7 \\ 9 & 0 & 4 & 2 \\ 3 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

which, together with its inverse

$$[Z]^{-1} = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 1.0 \\ 1.0 & -3.0 & -1.0 & 30.0 \\ 0.0 & 0.5 & -1.0 & -1.5 \\ 0.0 & -0.5 & 2.0 & -1.5 \end{bmatrix}$$

can be used to generate the modified modal matrix $[X] = [\Phi][Z]$ and its inverse $[X]^{-1} = [Z]^{-1}[\Phi]^{-1}$ which will reduce the modified dynamical matrix to the Jordan form

$$\begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

Example 2

As an example of a symmetric positive-definite system having distinct eigenvalues, consider the damped two story frame structure shown in Fig. 2 which has mass, damping, and stiffness matrices

$$[m] = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \quad [c] = \begin{bmatrix} 10 & -10 \\ -10 & 10 \end{bmatrix} \quad [k] = \begin{bmatrix} 500 & -240 \\ -240 & 200 \end{bmatrix}$$

respectively. The associated eigenvalues**

$$\mu_1 = \bar{\mu}_2 = -0.27974 - i\,5.5139$$

$$\mu_3 = \bar{\mu}_4 = -3.8869 - i\,14.722$$

** In this example problem, a bar over a complex quantity represents the complex conjugate of the quantity.

and the eigenvectors

$$\{\Phi\}_1 = \overline{\{\Phi\}_2} = \begin{Bmatrix} 0.35202 - i\,0.45261 \\ 0.67486 - i\,0.68635 \\ 0.07864 + i\,0.06783 \\ 0.11796 + i\,0.12834 \end{Bmatrix}$$

$$\{\Phi\}_3 = \overline{\{\Phi\}_4} = \begin{Bmatrix} 0.76649 - i\,1.0306 \\ -0.40953 + i\,1.1841 \\ 0.05259 - i\,0.06595 \\ -0.06832 + i\,0.04586 \end{Bmatrix}$$

which make up the columns of the modal matrix $[\Phi]$, constitute the complete solution of the original system.

This system will be modified by adding an equivalent viscous-damping force of 4 kip sec/in. between the first floor and ground. In matrix form this is incorporated by adding $\alpha\{p\}\lfloor p\rfloor$ to $[c]$ where $\alpha = 4$, and $\lfloor p\rfloor = [1 \quad 0]$. Since all four modification coefficients are nonzero, there are no unchanged eigenvalues, so all four eigenvalues must be obtained from the characteristic equation. They may be found to be

$$\lambda_1 = \bar{\lambda}_2 = -0.51056 - i\,5.4675$$

$$\lambda_3 = \bar{\lambda}_4 = -4.3229 - i\,14.686$$

and the corresponding eigenvectors become

$$\{Y\}_1 = \overline{\{Y\}_2} = \begin{Bmatrix} 0.33766 - i\,0.45382 \\ 0.63255 - i\,0.71917 \\ 0.07657 + i\,0.06891 \\ 0.11969 + i\,0.12687 \end{Bmatrix}$$

$$\{Y\}_3 = \overline{\{Y\}_4} = \begin{Bmatrix} -0.70909 + i\,1.0903 \\ 0.43177 - i\,1.1486 \\ -0.05524 - i\,0.06455 \\ 0.06401 + i\,0.04824 \end{Bmatrix}$$

This modified structure is identical to that analyzed by Berg.[6]

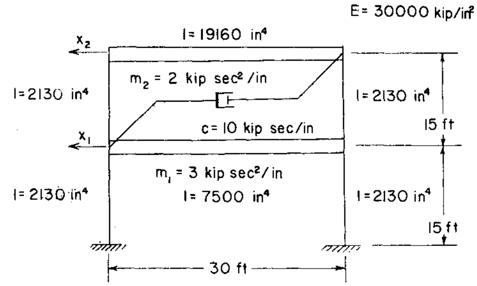

Fig. 2 Damped two-story frame.

Summary and Conclusions

The modification procedure presented here provides a means of generating a characteristic equation associated with a system obtained by modifying one of the system elements. Linear viscous-damping effects are incorporated by transforming the governing equations to $2n$ space, and the resulting effects of multiple eigenvalues are incorporated. A primary feature of this procedure is that the modified characteristic equation is of a form which is easily solved by existing numerical procedures. Another advantage is that multiple eigenvalues in a modified system are most likely to occur as unchanged eigenvalues. Thus, by removing these known eigenvalues prior to solution of the characteristic equation, the numerical difficulties associated with multiple eigenvalues are eliminated.

It often occurs in the analysis of large systems that the known solution of the original system characteristics consists of only a few of the lower modes and the associated eigenvalues. This is because the higher modes participate to a relatively minor extent in the system response, and are therefore not needed. When this situation occurs, the modification procedure may still be used, with only minor alterations, to generate an approximate solution of the modified system. Recall that for a system with n degrees of freedom, the matrices $[M]$ and $[K]$ are of dimension $2n$ by $2n$. Assume that for a symmetric positive-definite original system only m ($m < 2n$) eigenvalues and eigenvectors are known. The $2n$ by m partial modal matrix $[\Phi]$ may then be used to generate the m elements of the vector $\{U\}$ which are required for the formation of the modified characteristic equation. The only difference between the characteristic equation obtained in this approximation and that shown in Eq. (22) is the summation in the approximation consists of only m terms since the original system has essentially been approximated by one with m degrees of freedom. When the system is not symmetric and positive definite, a similar approximate solution may be generated provided that the partial solution of the original system consists not only of the m right eigenvectors $\{\Phi\}$ but also the m left eigenvectors $\lfloor\zeta\rfloor$.

References

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